Risk and Return

Topic 2

Gregory Lypny

Notation

Returns

$$\mathbf{R} = \begin{pmatrix} R_1 \\ R_2 \\ \vdots \\ R_j \\ \vdots \\ R_{n-1} \\ R_n \end{pmatrix}, \quad \bar{\mathbf{R}} = \begin{pmatrix} E(R_1) \\ E(R_2) \\ \vdots \\ E(R_j) \\ \vdots \\ E(R_{n-1}) \\ E(R_n) \end{pmatrix}, \quad R_f$$

Variance

$$\sigma^2 = E\left(R - E(R)\right)^2 = E\left(R^2\right) - \left(E(R)\right)^2$$

Covariance

$$\sigma_{ij} = E\left[\left(R_i - E\left(R_i\right)\right)\left(R_j - E\left(R_j\right)\right)\right] = E\left(R_i R_j\right) - E\left(R_i\right)E\left(R_j\right)$$

Correlation

$$\rho_{ij} = \frac{\sigma_{ij}}{\sigma_i \sigma_j}$$

Covariance matrix

$$\mathbf{V} = \begin{pmatrix} \sigma_{11} & \cdots & \sigma_{1n} \\ \vdots & \ddots & \vdots \\ \sigma_{n1} & \cdots & \sigma_{nn} \end{pmatrix}, \text{ where } \sigma_{ij} = \sigma_{ji}$$

Example: $$\mathbf{V} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$$

Wealth and investment weights

$$W_0 = \sum_j S_j = \mathbf{1}' \cdot \mathbf{S} = S_1 + \cdots + S_j + \cdots + S_n \Rightarrow \sum_j x_j = \mathbf{1}' \cdot \mathbf{x} = 1$$

$$\mathbf{S} = \begin{pmatrix} S_1 \\ \vdots \\ S_j \\ \vdots \\ S_n \end{pmatrix}, \quad \mathbf{x} = \frac{\mathbf{S}}{W_0} \begin{pmatrix} x_1 \\ \vdots \\ x_j \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

Income

$$y = \sum_j R_j \cdot S_j = \mathbf{R}' \cdot \mathbf{S} = \begin{pmatrix} R_1 & \cdots & R_n \end{pmatrix} \cdot \begin{pmatrix} S_1 \\ \vdots \\ S_n \end{pmatrix}$$

Data

The companies

File: S&P Big Cap 2008-12-31 to 2011-06-30.csv

Stocks: AAPL, DTV, KMB, NWSA, WAG

Dates: 30 Jan 2009 to 30 Jun 2011

N: 30 months

A Two-Asset Portfolio Frontier

The stocks

AAPL and DTV

$$R = \begin{pmatrix} 0.0489129 \\ 0.0296652 \end{pmatrix}$$

$$\overline{\sigma} = \begin{pmatrix} 0.0684982 \\ 0.0750907 \end{pmatrix}$$

$$V = \begin{pmatrix} 0.00469201 & 0.0024785 \\ 0.0024785 & 0.00563861 \end{pmatrix}$$

Portfolio expected return

$$E\left(R_p\right) = \mathbf{x}' \cdot \mathbf{R} = x_1 E\left(R_1\right) + x_2 E\left(R_2\right)$$
$$= x_1 E\left(R_1\right) + \left(1 - x_1\right) E\left(R_2\right)$$

or $x_1 = \dfrac{E\left(R_p\right) - E\left(R_2\right)}{E\left(R_1\right) - E\left(R_2\right)}$

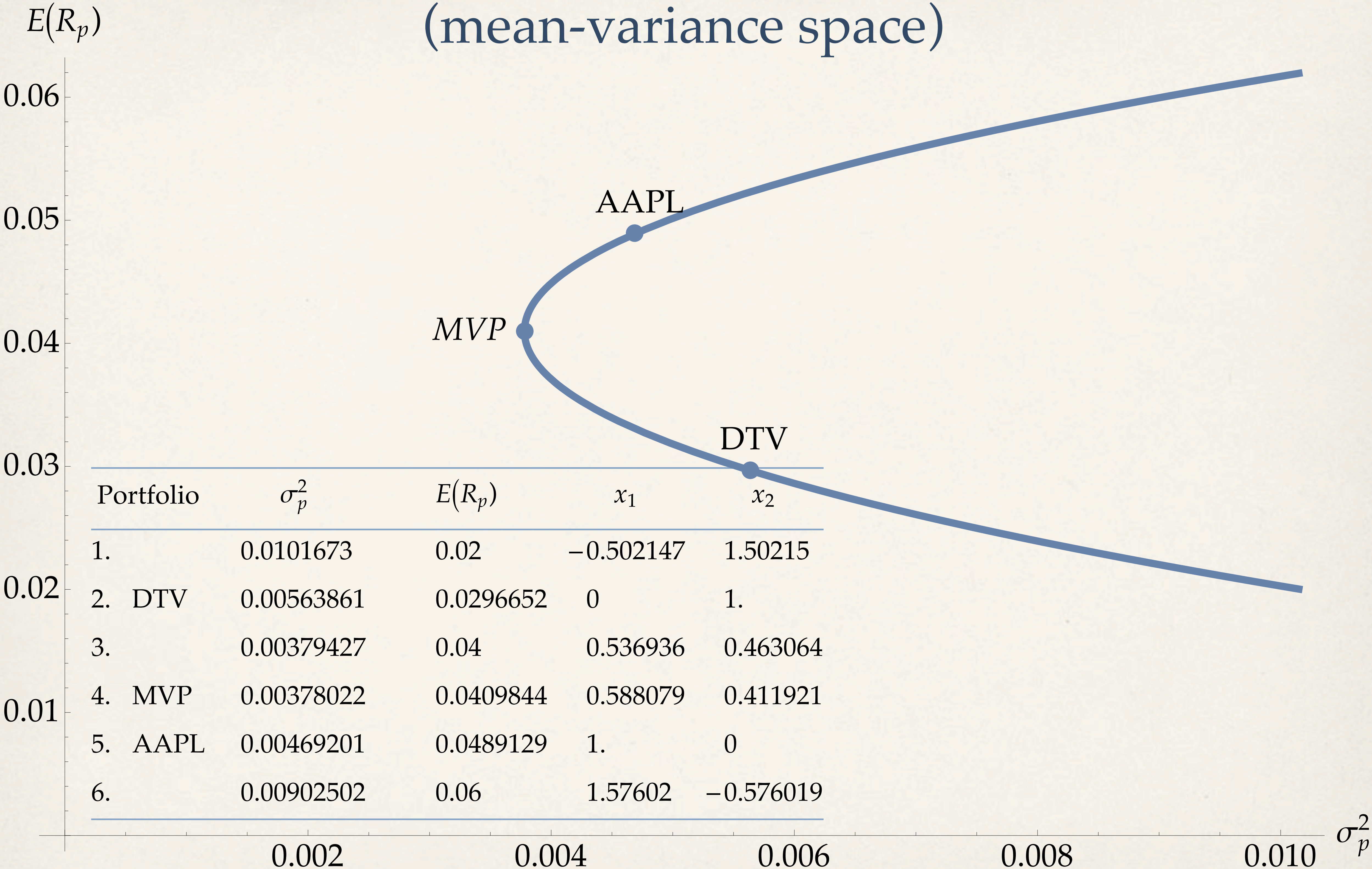
Portfolio	σ_p^2	σ_p	$E(R_p)$	x_1	x_2
1.	0.0101673	0.100833	0.02	−0.502147	1.50215
2. DTV	0.00563861	0.0750907	0.0296652	0	1.
3.	0.00379427	0.0615977	0.04	0.536936	0.463064
4. MVP	0.00378022	0.0614835	0.0409844	0.588079	0.411921
5. AAPL	0.00469201	0.0684982	0.0489129	1.	0
6.	0.00902502	0.0950001	0.06	1.57602	−0.576019

Portfolio variance

$$\sigma_p^2 = \mathbf{x}'.\mathbf{V}.\mathbf{x} = x_1^2\sigma_1^2 + x_2^2\sigma_2^2 + 2x_1x_2\sigma_{12}$$
$$= x_1^2\sigma_1^2 + \left(1 - x_1\right)^2\sigma_2^2 + 2x_1\left(1 - x_1\right)\sigma_1\sigma_2\rho_{12}$$

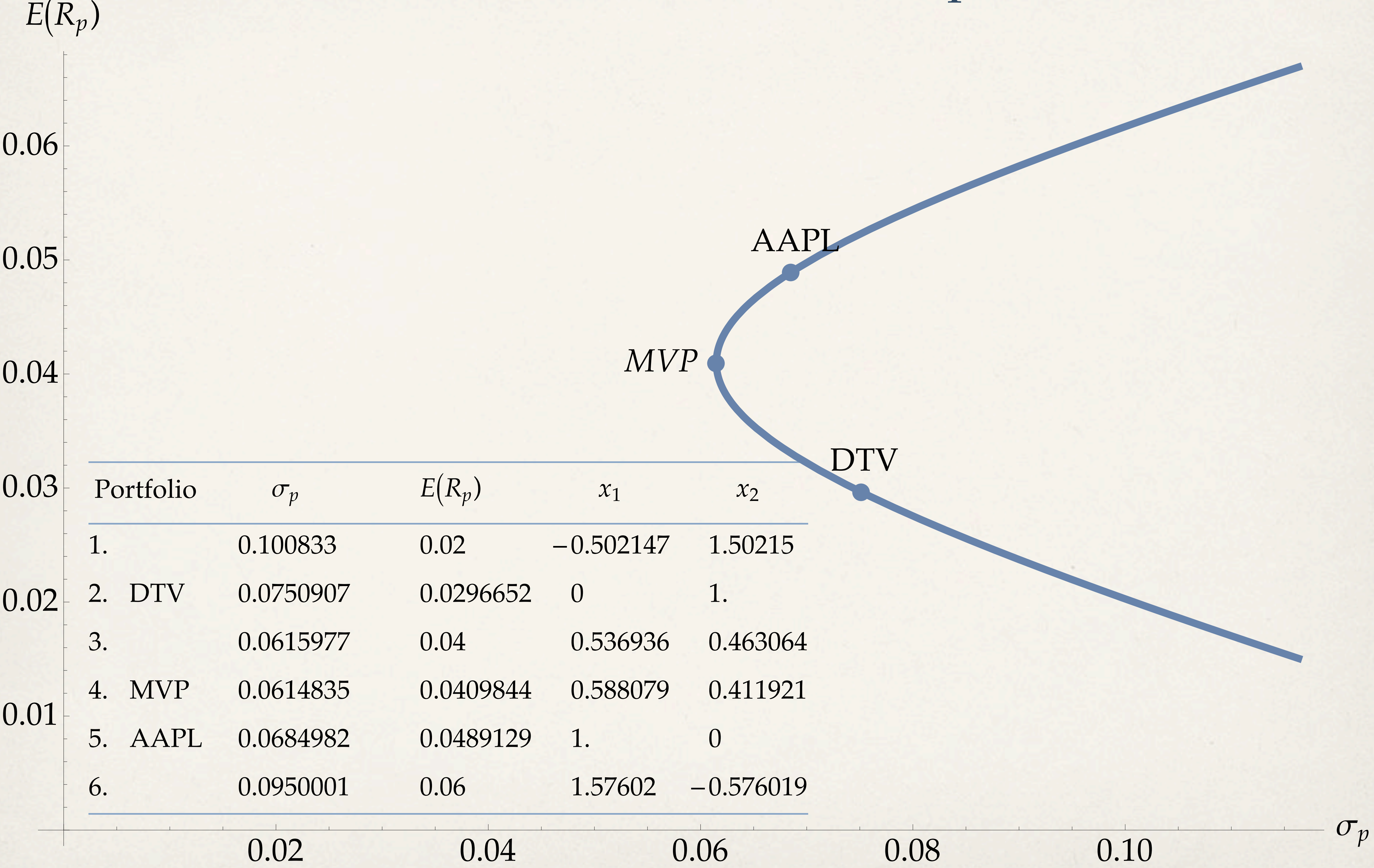
Portfolio	σ_p^2	σ_p	$E(R_p)$	x_1	x_2
1.	0.0101673	0.100833	0.02	−0.502147	1.50215
2. DTV	0.00563861	0.0750907	0.0296652	0	1.
3.	0.00379427	0.0615977	0.04	0.536936	0.463064
4. MVP	0.00378022	0.0614835	0.0409844	0.588079	0.411921
5. AAPL	0.00469201	0.0684982	0.0489129	1.	0
6.	0.00902502	0.0950001	0.06	1.57602	−0.576019

Minimum-Variance Frontier
(mean-variance space)



Portfolio	σ_p^2	$E(R_p)$	x_1	x_2
1.	0.0101673	0.02	−0.502147	1.50215
2. DTV	0.00563861	0.0296652	0	1.
3.	0.00379427	0.04	0.536936	0.463064
4. MVP	0.00378022	0.0409844	0.588079	0.411921
5. AAPL	0.00469201	0.0489129	1.	0
6.	0.00902502	0.06	1.57602	−0.576019

Minimum-Variance Frontier
(mean-standard deviation space)



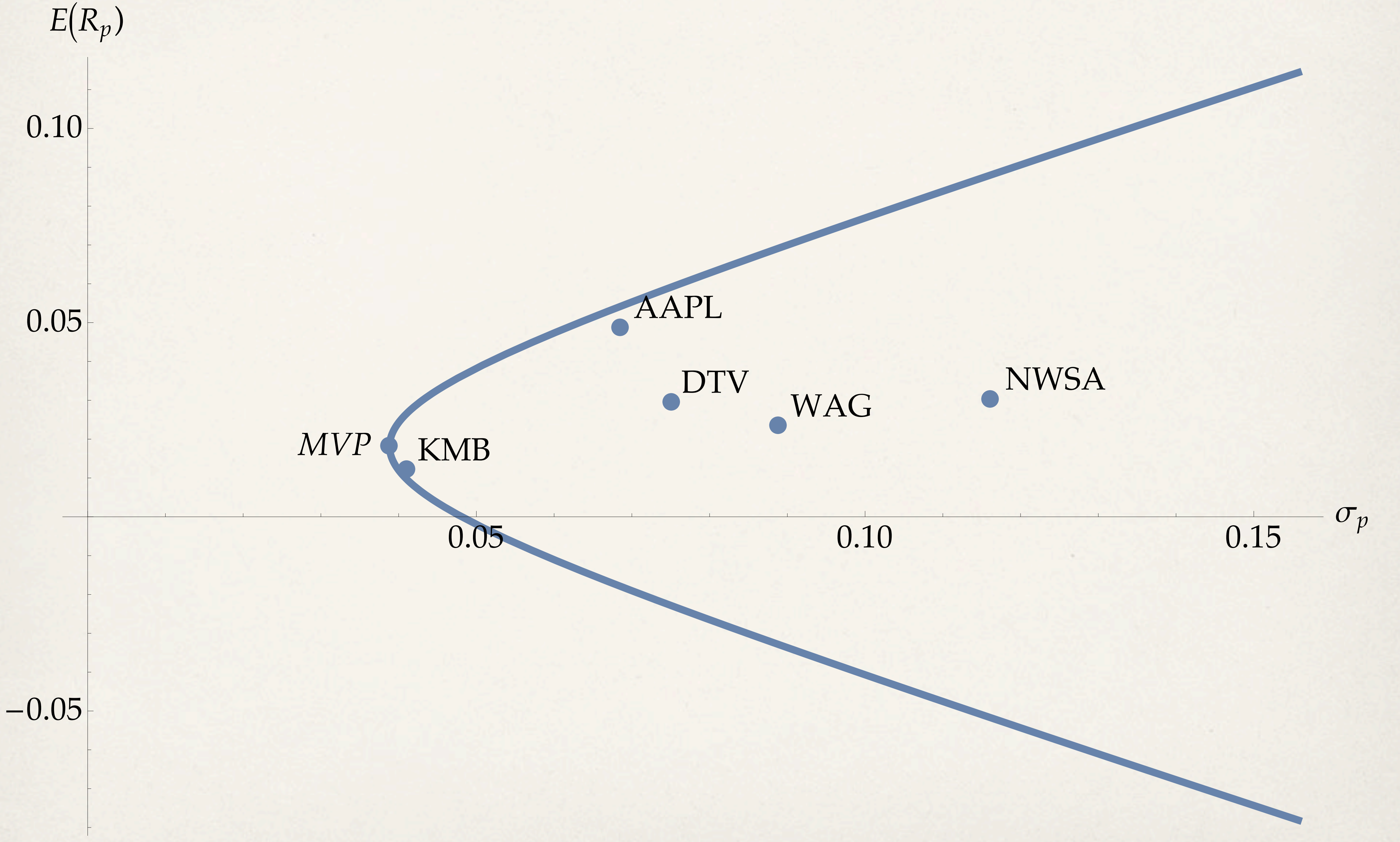
Portfolio		σ_p	$E(R_p)$	x_1	x_2
1.		0.100833	0.02	−0.502147	1.50215
2.	DTV	0.0750907	0.0296652	0	1.
3.		0.0615977	0.04	0.536936	0.463064
4.	MVP	0.0614835	0.0409844	0.588079	0.411921
5.	AAPL	0.0684982	0.0489129	1.	0
6.		0.0950001	0.06	1.57602	−0.576019

Special cases

- Derive the minimum-variance portfolio for a two-asset frontier
- Derive portfolio standard deviation for the case where the correlation of return between the two assets is
  - +1
  - -1

Portfolio Mathematics

The Goal



Optimal investment

$$\underset{<S>}{Min}\,\sigma_p^2 = \mathbf{S}'\cdot\mathbf{V}\cdot\mathbf{S} + \lambda_1\left[\bar{\mathbf{R}}'\cdot\mathbf{S} - \bar{y}\right] + \lambda_2\left[\mathbf{1}'\cdot\mathbf{S} - W_0\right]$$

$$\mathbf{S}: \quad 2\mathbf{VS} + \lambda_1\bar{\mathbf{R}} + \lambda_2\mathbf{1} = 0$$

$$\lambda_1: \quad \bar{\mathbf{R}}'\cdot\mathbf{S} - \bar{y}_i = 0$$

$$\lambda_2: \quad \mathbf{1}'\cdot\mathbf{S} - W_i = 0$$

$$\mathbf{S} = -\mathbf{V}^{-1}\frac{\left(\lambda_1\bar{\mathbf{R}} + \lambda_2\mathbf{1}\right)}{2} = \mathbf{V}^{-1}\begin{pmatrix}\bar{\mathbf{R}} & \mathbf{1}\end{pmatrix}\begin{pmatrix}-\lambda_1/2 \\ -\lambda_2/2\end{pmatrix}$$

Optimal investment {continued}

$$\mathbf{S} = -\mathbf{V}^{-1}\frac{\left(\lambda_1\bar{\mathbf{R}} + \lambda_2\mathbf{1}\right)}{2} = \mathbf{V}^{-1}\begin{pmatrix} \bar{\mathbf{R}} & \mathbf{1} \end{pmatrix}\begin{pmatrix} -\lambda_1/2 \\ -\lambda_2/2 \end{pmatrix}$$

$$\mathbf{X} = \frac{\mathbf{S}}{W_0} = \mathbf{V}^{-1}\begin{pmatrix} \bar{\mathbf{R}} & \mathbf{1} \end{pmatrix}\mathbf{Q}^{-1}\begin{pmatrix} \bar{R}_p \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} \bar{\mathbf{R}} & \mathbf{1} \end{pmatrix}'\mathbf{S} = \begin{pmatrix} \bar{\mathbf{R}} & \mathbf{1} \end{pmatrix}'\mathbf{V}^{-1}\begin{pmatrix} \bar{\mathbf{R}} & \mathbf{1} \end{pmatrix}\begin{pmatrix} -\lambda_1/2 \\ -\lambda_2/2 \end{pmatrix} = \mathbf{Q}\begin{pmatrix} -\lambda_1/2 \\ -\lambda_2/2 \end{pmatrix}$$

$$\text{where } \mathbf{Q} = \begin{pmatrix} \bar{\mathbf{R}}'\mathbf{V}^{-1}\bar{\mathbf{R}} & \bar{\mathbf{R}}'\mathbf{V}^{-1}\mathbf{1} \\ \mathbf{1}'\mathbf{V}^{-1}\bar{\mathbf{R}} & \mathbf{1}'\mathbf{V}^{-1}\mathbf{1} \end{pmatrix} = \begin{pmatrix} \bar{\mathbf{R}}'\mathbf{V}^{-1}\bar{\mathbf{R}} & \bar{\mathbf{R}}'\mathbf{V}^{-1}\mathbf{1} \\ \mathbf{1}'\mathbf{V}^{-1}\bar{\mathbf{R}} & \mathbf{1}'\mathbf{V}^{-1}\mathbf{1} \end{pmatrix} = \begin{pmatrix} B & A \\ A & C \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -\lambda_1/2 \\ -\lambda_2/2 \end{pmatrix} = \mathbf{Q}^{-1}\begin{pmatrix} \bar{\mathbf{R}} & \mathbf{1} \end{pmatrix}'\mathbf{S} = \mathbf{Q}^{-1}\begin{pmatrix} \bar{y} \\ W \end{pmatrix}$$

Optimal investment {*finis*}

$$\mathbf{x} = \frac{\mathbf{S}}{W_0} = \mathbf{V}^{-1}\begin{pmatrix} \bar{\mathbf{R}} & \mathbf{1} \end{pmatrix}\mathbf{Q}^{-1}\begin{pmatrix} \bar{R}_p \\ 1 \end{pmatrix}$$

$$\mathbf{x} = \mathbf{V}^{-1}\begin{pmatrix} \bar{\mathbf{R}} & \mathbf{1} \end{pmatrix}\frac{\begin{pmatrix} C & -A \\ -A & B \end{pmatrix}}{D}\begin{pmatrix} \bar{R}_p \\ 1 \end{pmatrix}$$

$$\mathbf{x} = \frac{1}{D}\left[B\left(\mathbf{V}^{-1}\mathbf{1}\right) - A\left(\mathbf{V}^{-1}\bar{\mathbf{R}}\right)\right] + \frac{1}{D}\left[C\left(\mathbf{V}^{-1}\bar{\mathbf{R}}\right) - A\left(\mathbf{V}^{-1}\mathbf{1}\right)\right]\bar{R}_p$$

$\mathbf{x} = \mathbf{g} + \mathbf{h}\bar{R}_p$ Optimal weights are linear in expected portfolio return

Minimum-variance frontier

$$\sigma^2 = \mathbf{S'VS} = \mathbf{S'V}\left(\mathbf{V}^{-1}\begin{pmatrix} \bar{\mathbf{R}} & \mathbf{1} \end{pmatrix}\mathbf{Q}^{-1}\begin{pmatrix} \bar{y} \\ W_0 \end{pmatrix}\right)$$

$$\therefore \sigma_p^2 = \left(\mathbf{S'}\begin{pmatrix} \bar{\mathbf{R}} & \mathbf{1} \end{pmatrix}\right)\mathbf{Q}^{-1}\begin{pmatrix} \bar{y} \\ W_0 \end{pmatrix} = \begin{pmatrix} \bar{y} & W_0 \end{pmatrix}\mathbf{Q}^{-1}\begin{pmatrix} \bar{y} \\ W_0 \end{pmatrix}$$

$$\therefore \sigma_p^2 = \frac{\sigma^2}{W_0^2} = \begin{pmatrix} \bar{R}_p & 1 \end{pmatrix}\mathbf{Q}^{-1}\begin{pmatrix} \bar{R}_p \\ 1 \end{pmatrix}$$

$$\therefore \sigma_p^2 = \frac{1}{D}\left(C\bar{R}_p^2 - 2A\bar{R}_p + B\right)$$

Parabola in mean-variance space
Hyperbola in mean-standard deviation space

Minimum-variance portfolio

$$\sigma_p^2 = \frac{\sigma^2}{W^2} = \begin{pmatrix} \bar{R}_p & 1 \end{pmatrix} \mathbf{Q}^{-1} \begin{pmatrix} \bar{R}_p \\ 1 \end{pmatrix}$$

$$\sigma_p^2 = \frac{1}{D}\left(C\bar{R}_p^2 - 2A\bar{R}_p + B\right)$$

$$\frac{d\sigma_p^2}{d\bar{R}_p} = 0 \Rightarrow MVP : \left(\sigma_p^2, \bar{R}_p\right) = \left(\frac{1}{C}, \frac{A}{C}\right)$$

Orthogonal portfolios {finding uncorrelated portfolios}

$$\operatorname{cov}(y_k, y_u) = \mathbf{S_k' V S_u} = \left(\begin{pmatrix} \bar{y}_k & W_0 \end{pmatrix} \mathbf{Q}^{-1} \begin{pmatrix} \bar{y}_u \\ W_0 \end{pmatrix} \right)$$

$$\therefore \sigma_{ku} = \operatorname{cov}(\bar{R}_k, \bar{R}_u) = \frac{1}{D} \left(\begin{pmatrix} \bar{R}_k & 1 \end{pmatrix} \begin{pmatrix} C & -A \\ -A & B \end{pmatrix} \begin{pmatrix} \bar{R}_u \\ 1 \end{pmatrix} \right)$$

$$\therefore \sigma_{ku} = \frac{1}{D} C\bar{R}_k \bar{R}_u - A\bar{R}_k - A\bar{R}_u + B$$

$$\therefore \sigma_{ku} = 0 \Rightarrow \bar{R}_u = \frac{A\bar{R}_k - B}{C\bar{R}_k - A}$$ We'll use this later

A Five-Asset Portfolio Frontier

Means and standard deviations

$$
\begin{array}{l}
\text{AAPL} \\
\text{DTV} \\
\text{KMB} \\
\text{NWSA} \\
\text{WAG}
\end{array}
\qquad
\overline{R} = \begin{pmatrix} 0.0489129 \\ 0.0296652 \\ 0.0122415 \\ 0.0304791 \\ 0.0235527 \end{pmatrix}
\qquad
\overline{\sigma} = \begin{pmatrix} 0.0684982 \\ 0.0750907 \\ 0.0410235 \\ 0.116067 \\ 0.0888068 \end{pmatrix}
$$

Covariances

$$V = \begin{pmatrix} 0.00469201 & 0.0024785 & 0.00120074 & 0.00474886 & 0.00374848 \\ 0.0024785 & 0.00563861 & 0.00102798 & 0.00369911 & 0.00253769 \\ 0.00120074 & 0.00102798 & 0.00168292 & 0.00199206 & 0.00136948 \\ 0.00474886 & 0.00369911 & 0.00199206 & 0.0134715 & 0.00357577 \\ 0.00374848 & 0.00253769 & 0.00136948 & 0.00357577 & 0.00788664 \end{pmatrix}$$

$$V^{-1} = \begin{pmatrix} 0.00469201 & 0.0024785 & 0.00120074 & 0.00474886 & 0.00374848 \\ 0.0024785 & 0.00563861 & 0.00102798 & 0.00369911 & 0.00253769 \\ 0.00120074 & 0.00102798 & 0.00168292 & 0.00199206 & 0.00136948 \\ 0.00474886 & 0.00369911 & 0.00199206 & 0.0134715 & 0.00357577 \\ 0.00374848 & 0.00253769 & 0.00136948 & 0.00357577 & 0.00788664 \end{pmatrix}$$

Portfolio math parameters

$$\mathbf{Q} = \begin{pmatrix} \bar{\mathbf{R}}' \cdot \mathbf{V}^{-1} \cdot \bar{\mathbf{R}} & \bar{\mathbf{R}}' \cdot \mathbf{V}^{-1} \cdot \mathbf{1} \\ \mathbf{1}' \cdot \mathbf{V}^{-1} \cdot \bar{\mathbf{R}} & \mathbf{1}' \cdot \mathbf{V}^{-1} \cdot \mathbf{1} \end{pmatrix}$$

$$= \begin{pmatrix} B & A \\ A & C \end{pmatrix} = \begin{pmatrix} 0.625609 & 12.0295 \\ 12.0295 & 663.12 \end{pmatrix}$$

$$\mathbf{Q}^{-1} = \frac{\begin{pmatrix} C & -A \\ -A & B \end{pmatrix}}{D} = \begin{pmatrix} 2.45467 & -0.0445295 \\ -0.0445295 & 0.00231582 \end{pmatrix}$$

$$D = BC - A^2 = 270.146$$

Optimal weights

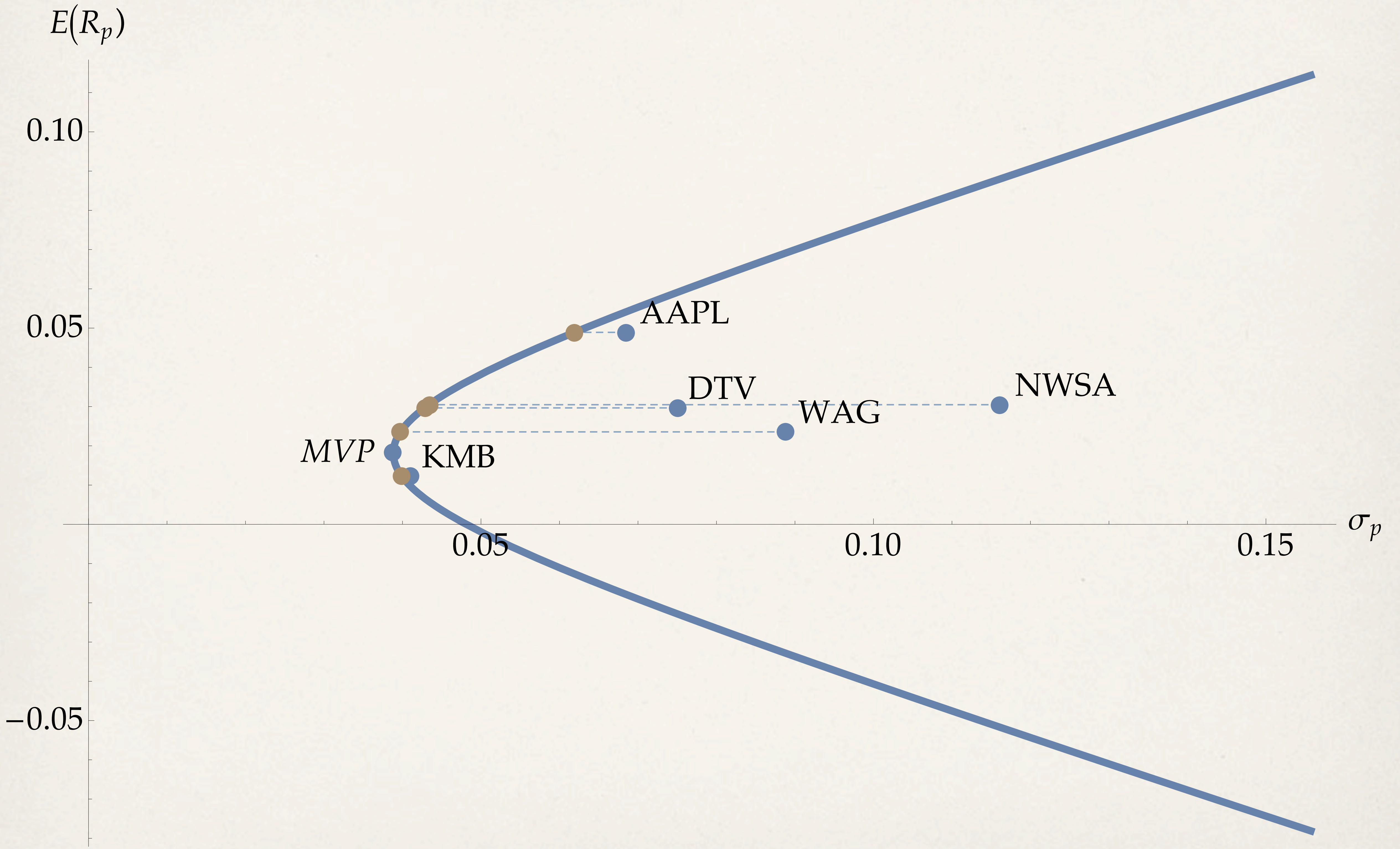
$$\mathbf{x} = \mathbf{g} + \mathbf{h}\bar{R}_p = \begin{pmatrix} -0.404393 \\ 0.0934881 \\ 1.21151 \\ -0.0238107 \\ 0.123201 \end{pmatrix} + \begin{pmatrix} 31.1317 \\ 1.29537 \\ -20.2935 \\ -3.89382 \\ -8.23981 \end{pmatrix} \bar{R}_p$$

$$\sigma_p = \sqrt{\frac{1}{D}\left(C\bar{R}_p^2 - 2A\bar{R}_p + B\right)}$$
$$= \sqrt{\frac{1}{270.146}\left(663.12\times\bar{R}_p^2 - 2\times 12.0295\times\bar{R}_p + 0.625609\right)}$$
$$= \sqrt{\mathbf{x}\cdot\mathbf{V}\cdot\mathbf{x}}$$

$$\mathbf{x} = \mathbf{g} + \mathbf{h}\bar{R}_p = \begin{pmatrix} -0.404393 \\ 0.0934881 \\ 1.21151 \\ -0.0238107 \\ 0.123201 \end{pmatrix} + \begin{pmatrix} 31.1317 \\ 1.29537 \\ -20.2935 \\ -3.89382 \\ -8.23981 \end{pmatrix}\bar{R}_p$$

Portfolio	σ_p	$E(R_p)$	x_1	x_2	x_3	x_4	x_5
1.	0.0408741	0.01	−0.0930757	0.106442	1.00858	−0.0627488	0.0408028
2. KMB	0.039918	0.0122415	−0.0232947	0.109345	0.963093	−0.0714767	0.0223334
3. MVP	0.0388333	0.0181407	0.160358	0.116987	0.843377	−0.0944472	−0.0262749
4. WAG	0.0397482	0.0235527	0.328843	0.123997	0.733549	−0.115521	−0.0708689
5. DTV	0.0428257	0.0296652	0.519136	0.131915	0.609505	−0.139321	−0.121235
6.	0.0430495	0.03	0.529559	0.132349	0.60271	−0.140625	−0.123993
7. NWSA	0.0433787	0.0304791	0.544476	0.13297	0.592987	−0.142491	−0.127941
8. AAPL	0.0619066	0.0489129	1.11835	0.156848	0.218901	−0.214269	−0.279832
9.	0.0762175	0.06	1.46351	0.17121	−0.00609397	−0.25744	−0.371188

$$MVP:\begin{pmatrix} \sigma & \mathrm{E}(R) \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{C}} & \frac{A}{C} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{663.12}} & \frac{12.0295}{663.12} \end{pmatrix}$$


$E(R_p)$
0.10
0.05
−0.05
σ_p
0.05
0.10
0.15
AAPL
DTV
NWSA
WAG
MVP
KMB

Capital Market Line {next week}